NUMERICAL AND ANALYTICAL TREATMENT FOR THE EFFECTIVENESS OF OPERATOR METHOD FOR A SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract

Fractional calculus had shown to be adequate models for various areas of engineering, science, finance, applied mathematics, bio-engineering, and others. However, many researchers remain unaware of this field. In this paper, we deal with finding an analytical criterion to determine the existence or non-existence of a numerical solution for a system of fractional ordinary differential equations, which is able to be reduced to a system of ordinary differential equations with integer derivatives in terms of exp-function method, which was first proposed by He and Wu under the space of functions $\mathcal{C}$. The fractional derivative is described in the Riemann-Liouville sense.

Our study will go across some stages. The first stage, we present how to reduce a system of fractional ordinary differential equations to a system of ordinary differential equations with integer derivatives by using the properties of class $\mathcal{C}$. The second stage, we introduce a theorem and its proof, which gives an

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operator method to express the solution of a system of ordinary differential equations as a generalization of Navickas work in this field. The final stage, we present an algorithm, which is used to clarify a criterion to express the solution of fractional differential equation under the space Ζ in terms of exponential functions, and we give three examples of systems of fractional differential equations in which two of them are linear and the third is non-linear to be solved by using our proposed technique.

1. Introduction

Fractional differential equations have a wide existence in different fields ([1], [3]). There are many approximate methods were presented in the field of fractional differential equations ([8], [11]), such as the homotopy analysis method [5], the Adomian decomposition method [20], and also numerical methods, such as the finite difference method [21].

There are many papers are concerned with finding solutions of fractional differential equations ([4], [9], [10], [13], [14], [22], [23]). The exp-function method was proposed by He and Wu in 2006 to seek solitary solutions, periodic solutions, and compacton-like solutions of non-linear differential equations ([6], [24], [25]).

Hua presented the exponential function rational method [7]. Navickas and Ragulskis constructed an analytical criterion determining, if there exists a solution of a differential equation, which can be found by the exp-function method. The employment of this criterion else gives the structure of the solution. We will adapt this criterion to determine, if a solution of some systems of fractional differential equations under a special space of functions Ζ. In this paper, we are concerned with finding the components of the vector solution \((y_1(t), y_2(t), \ldots, y_n(t))\) for the system of fractional differential equations

\[ D^\alpha y_i(t) = f_i(t; y_1, y_2, \ldots, y_n), \quad i = 1, 2, \ldots, n, \quad \alpha \in (0, 1), \]  \hspace{1cm} (1)

subject to the initial conditions

\[ y_i(t) \big|_{t=v} = c_i, \quad v \in \mathbb{R}. \]  \hspace{1cm} (2)
Under the space of functions $\mathcal{C}$, we will provide good quality algorithms for the solution of the systems (1)-(2). If the system is linear, we provide an exact solution after reducing it to a system of ordinary differential equations with integer derivatives by using the properties of class $\mathcal{C}$. But if it is a non-linear system, we provide an approximate solution by using the properties of class $\mathcal{C}$ and an approximate expression of the fractional Leibnitz rule.

In [17], Navickas presented a criterion depends on the definition of the Hankel rank, Hankel characteristic equation and its solutions. The main goal of this paper is to extend this work to specify the possibility of the expression of the solution of a system of differential equations in terms of exponential functions.

2. The Space of Functions $\mathcal{C}$

**Definition 1.** Let $\text{Re}(\nu) > 0$ and let $f(t)$ be piecewise continuous on $(0, \infty)$ and integrable on any finite subinterval of $J = [0, \infty)$. Then for $t > 0$, we call [19]

$$0D^{\nu}_{t}f(t) := \frac{1}{\Gamma(\nu)} \int_{0}^{t} \frac{f(\zeta)}{(t-\zeta)^{\nu-1}} d\zeta,$$

(3)

where $\Gamma$ is the Gamma function.

We shall denote by $\mathcal{C}$ the class of functions described in Definition 1. There are some functions like $f(t)$ in which the fractional derivative $D^\mu f(t)$ of order $\mu$ may be obtained from the fractional integral $D^{-\nu} f(t)$ by replacing $\nu$ by $-\mu$. In other words, this conclusion is achieved

$$D^\mu f(t) = D^{-\nu} f(t), \quad \nu = -\mu.$$

(4)

This conclusion is not necessarily achieved for all functions of class $\mathcal{C}$. Motivated by these observations, we define a space of functions $\mathcal{C}$ as a subclass of $\mathcal{C}$ such that $f(t)$ has both a fractional integral and a fractional derivative of any order.
Definition 2. The space of functions $\mathcal{C}$ is a subclass of $C$ in which all functions have the form

$$t^\lambda \eta(t),$$

$$t^\lambda \ln(t)\eta(t),$$

where $\lambda > -1$, $\eta(t) = \sum_{n=0}^{\infty} a_n t^n$ has a radius of convergence $R$, and $t \geq R$.

Properties of the class $\mathcal{C}$

(1) Leibnitz’s formula for fractional derivatives

$$D^\mu [t^p f(t)] = \sum_{r=0}^{p} \left( \frac{\mu}{r} \right) [D^r t^p] [D^{\mu-r} f(t)], \quad \mu > 0, \quad p > 0, \quad f(t) \in \mathcal{C},$$

provided that $p$ is a positive integer, $f(t)$ is of class $\mathcal{C}$ and $t > 0$.

Comparing with the following formula:

$$D^{-\nu} [t^p f(t)] = \sum_{k=0}^{p} \left( \frac{-\nu}{k} \right) [D^k t^p] [D^{(-\nu+k)} f(t)].$$

It is obvious that they are identical with $\nu$ replaced by $\mu$ except that Equation (7) is valid only inside the class $\mathcal{C}$, but Equation (8) is valid for a wider class, namely, $C$ in which $\mathcal{C}$ is a subclass.

As a special case from Equation (7) when $p = 1$

$$D^\mu [t f(t)] = t D^\mu f(t) + \mu D^{-1} f(t), \quad \mu > 0.$$  

(2) The law of exponents for fractional derivatives.

If $\mu, \nu > 0$ and $f(t)$ is continuous on $[0, \infty)$, we introduce the law of exponents for fractional integrals

$$D^{-\mu} [D^{-\nu} f(t)] = D^{-(\mu+\nu)} f(t) = D^{-\nu} [D^{-\mu} f(t)],$$
but this formula cannot be extended to fractional derivatives by replacing $-\mu, -\nu$ with $\mu, \nu$, respectively, without some additional restrictions on $f(t)$.

Illustrative example:

If $f(t) = t^{1/2}$, $u = 1/2$, $v = 3/2$, then $D^u[D^v t^{1/2}] = D^{1/2}[D^{3/2} t^{1/2}] = 0$,

If $D^u+v t^{1/2} = D^2 t^{1/2} = -\frac{1}{4} t^{-3/2}$, then $D^{u+v} t^{1/2} \neq D^u[D^v t^{1/2}]$.

For these additional restrictions, we introduce the following theorem:

**Theorem 1.** Let $f(t) \in \mathcal{C}$, $m$ is the smallest integer greater than or equal to $u$ such that $u \geq \lambda + 1$, $n$ is the smallest integer greater than or equal to $v$ such that $v \geq \lambda + 1$, then

$$D^v[D^u f(t)] = D^u[D^v f(t)] = D^{u+v} f(t), \quad (11)$$

if

(a) $u < \lambda + 1$ and $v < \lambda + 1$;

(b) $u < \lambda + 1$ and $v \geq \lambda + 1$ provided that $a_0 = a_1 = \ldots = a_{n-1} = 0$;

(c) $u \geq \lambda + 1$ and $v < \lambda + 1$ provided that $a_0 = a_1 = \ldots = a_{m-1} = 0$;

(d) $u \geq \lambda + 1$ and $v \geq \lambda + 1$ provided that $a_0 = a_1 = \ldots = a_{p-1} = 0$;

such that $p = \max(m, n)$.

(3) $D^\mu f(t) = [D^{-v} f(t)](\nu = -\mu)$, $\mu$ is an arbitrary.

So in this section, we introduced the space of functions $\mathcal{C}$, which is a subclass of the class $\mathcal{C}$ which possesses some properties more than the wider class $\mathcal{C}$.

For more details about the space of functions $\mathcal{C}$, see [12].
3. Reduction of a System of Ordinary Fractional Differential Equations under the Space of Functions $\mathcal{C}$

In this section, we introduce two examples of linear and non-linear systems of fractional differential equations, which can be reduced by using the properties of the space of functions $\mathcal{C}$ to systems of ordinary differential equations with integer derivatives.

The linear system will be reduced exactly using the properties of the space of functions $\mathcal{C}$. But the non-linear system will be reduced approximately using the following definition in addition to the properties of the space of functions $\mathcal{C}$.

Also, in this section, we are concerned with providing good quality algorithm for the solution of a system of fractional differential equations (FDE) of the following general form:

$$ t^n D^\alpha y_i(t) = f_i(t; y_1, y_2, \ldots, y_n), \quad i = 1, 2, \ldots, n, \quad \alpha \in (0, 1), \quad n \in \mathbb{N}, \tag{12} $$

subject to the initial conditions, $y_i(t) |_{t=0} = c_i, \quad v \in \mathbb{R}$.

**Definition 3.** If $f(\tau)$ is continuous in $[a, t]$ and $\varphi(\tau)$ has $n+1$ continuous derivatives in $[a, t]$, then the fractional derivative of the product $\varphi(t)f(t)$ is given by [19]

$$ a D_t^\mu [\varphi(t)f(t)] = \sum_{k=0}^{n} \binom{\mu}{k} [\varphi^{(k)}(t)] [a D_t^{\mu-k}f(t)] - R_n^a(t), \tag{13} $$

where $n \geq \mu + 1$; and

$$ R_n^a(t) = \frac{(-1)^n(t-a)^{n+1}}{n! \Gamma(-\mu)} \int_0^1 \int_0^1 F_a(t, \zeta, \eta) d\eta d\zeta, $$
The sum in (13) can be considered as a partial sum of an infinite series and a remainder of that series. If we neglect the remainder part \( R_n(t) \), then the fractional derivative of the product \( \varphi(t)f(t) \) is approximately given by

\[
a D^\mu_t [\varphi(t)f(t)] = \sum_{k=0}^{n} \binom{\mu}{k} \varphi^{(k)}(t) \left[ a D^{\mu-k}_t f(t) \right], \quad n \geq \mu + 1. \tag{14}
\]

**Example 3.1.** Consider the linear system of fractional differential equations of the form

\[
D^{\frac{1}{2}} y_1(t) = y_2(t), \tag{15}
\]
\[
D^{\frac{1}{2}} y_2(t) = 2y_1(t) - y_2(t). \tag{16}
\]

Operate on both sides of (15) and (16) by \( D^{\frac{1}{2}} \), we obtain

\[
y_1'(t) = D^{\frac{1}{2}} y_2(t), \tag{17}
\]
\[
y_2'(t) = 2D^{\frac{1}{2}} y_1(t) - D^{\frac{1}{2}} y_2(t). \tag{18}
\]

Add (17) and (18) to give

\[
D^{\frac{1}{2}} y_1(t) = \frac{1}{2} (y_1' + y_2'). \tag{19}
\]

Substitute by (17), (19) in (15), (16), respectively, we obtain

\[
y_1' = 2y_1 - y_2, \tag{20}
\]
\[
y_2' = 3y_2 - 2y_1. \tag{21}
\]
so, the linear system of fractional differential equations (15)-(16) reduced to the linear system of ordinary differential equations with integer derivatives (20)-(21).

**Example 3.2.** Consider the non-linear system of fractional differential equations of the form

\[ x \frac{D^\frac{1}{2}}{} y_1(t) = y_1^2(t), \]  
\[ x \frac{D^\frac{1}{2}}{} y_2(t) = -y_1(t) + y_2^2(t). \]  

Operate on both sides of (22) and (23) by \( \frac{D^\frac{1}{2}}{} \), we obtain

\[ xy_1'(t) + \frac{1}{2} y_1(t) = \sum_{k=0}^{2} \left( \frac{1}{2} \right)^k y_1^{(k)}(t)y_2^{(\frac{1}{2} - k)}(t) \]
\[ = \left( y_1(t) + \frac{3}{8} \right) \frac{D^\frac{1}{2}}{} y_1(t), \]  
\[ xy_2'(t) + \frac{1}{2} y_2(t) = -\frac{D^\frac{1}{2}}{} y_1(t) - \left( y_2(t) + \frac{3}{8} \right) \frac{D^\frac{1}{2}}{} y_2(t). \]  

Substitute by (24), (25) in (22), (23), respectively, we get

\[ y_1' = \frac{1}{x^2} \left[ y_1^2 (y_1 + \frac{3}{8}) - \frac{x}{2} y_1 \right], \]  
\[ y_2' = \frac{1}{x^2} \left[ (y_2^2 - y_1^2) (y_2 + \frac{3}{8}) - y_1 - \frac{x}{2} y_2 \right], \]  

so, the non-linear system of fractional differential equations (22)-(23) can be reduced approximately to the non-linear system of ordinary differential equations with integer derivatives (26)-(27).
4. Operator Representation for the Solution of a System of ODEs

**Definition 4.** The linear operator

\[ D_{st} = F_1(x; s, t)D_s + F_2(x; s, t)D_t, \]

is called a *generalized operator* of differentiation such that \( F_1(x; s, t) \) and \( F_2(x; s, t) \) are polynomials with respect to the variables \( x, s, \) and \( t. \)

**Definition 5.** The linear operator

\[ G(D_{st}) = \sum_{k=0}^{\infty} (L_x D_{st})^k, \]

is called the *generalized multiplicative operator*, such that \( L_x \) is the usual linear operator of integration, \( (L_x D_{st})^0 = 1 \) and it possesses some properties ([2], [16]).

For more details about these operators and their properties, see ([15], [16]).

In the following theorem, we start with a system of ODEs and obtain its operator representation:

**Theorem 2.**

\[ x^{(n)} = P(t; x, x', \ldots, x^{(n-1)}; y, y', \ldots, y^{(n-1)}); \quad (28) \]

\[ y^{(n)} = Q(t; x, x', \ldots, x^{(n-1)}; y, y', \ldots, y^{(n-1)}); \quad (29) \]

under the initial conditions

\[ x(t; s_1, s_2, \ldots, s_n; r_1, r_2, \ldots, r_n; v)|_{t=u} = s_1 \text{ and } y(t; s_1, s_2, \ldots, s_n; r_1, r_2, \ldots, r_n; v)|_{t=u} = r_1, \]

where \( v \in \mathbb{R}; \, k = 1, \, 2, \, \ldots, \, n - 1 \).

The solution of this system has the following operator representation:

\[
x(t; \, s_1, \, s_2, \ldots, \, s_n; \, r_1, \, r_2, \ldots, \, r_n; \, v) = \sum_{k=0}^{\infty} \frac{(t - v)^k}{k!} [D]^k s_1,
\]

\[
y(t; \, s_1, \, s_2, \ldots, \, s_n; \, r_1, \, r_2, \ldots, \, r_n; \, v) = \sum_{k=0}^{\infty} \frac{(t - v)^k}{k!} [D]^k r_2,
\]

where \( D \) is the general operator of differentiation

\[
D = D_v + s_2 D_{s_1} + s_3 D_{s_2} + \ldots + s_n D_{s_{n-1}} + PD_{s_n} + r_2 D_{r_1} + r_3 D_{r_2} + \ldots + r_n D_{r_{n-1}} + QD_{r_n}.
\]

**Proof.** Assume that

\[
z(t; \, s_1, \, s_2, \ldots, \, s_n; \, r_1, \, r_2, \ldots, \, r_n; \, v) = G(D)s_1,
\]

\[
w(t; \, s_1, \, s_2, \ldots, \, s_n; \, r_1, \, r_2, \ldots, \, r_n; \, v) = G(D)r_1,
\]

then

\[
z^{(n-1)}(t; \, s_1, \, s_2, \ldots, \, s_n; \, r_1, \, r_2, \ldots, \, r_n) = D^{(n-1)}_v G(D)s_1 = D^{(n-1)}_{(v; s_1, \ldots, s_n; r_1, \ldots, r_n)} G(D)s_1 = G(D)D^{(n-1)}_{(v; s_1, \ldots, s_n; r_1, \ldots, r_n)} s_1 = G(D)s_n,
\]

\[
z^{(n)}(t; \, s_1, \, s_2, \ldots, \, s_n; \, r_1, \, r_2, \ldots, \, r_n) = D_v G(D)s_n = D_{(v; s_1, \ldots, s_n; r_1, \ldots, r_n)} G(D)s_n = G(D)D_{(v; s_1, \ldots, s_n; r_1, \ldots, r_n)} s_n = G(D)P(v; \, s_1, \, s_2, \ldots, \, s_n; \, r_1, \, r_2, \ldots, \, r_n) = P(v + t; \, z, \, z', \ldots, \, z^{(n)}; \, w, \, w', \ldots, \, w^{(n)})
\]
Using the same steps, we obtain
\[
 w^{(n)}(t - v; s_1, s_2, \ldots, s_n; r_1, r_2, \ldots, r_n) = Q(t; z, z', \ldots, z^{(n)}, w, w', \ldots, w^{(n)}).
\]  
(31)

Compare (30), (31) with (28), (29), respectively, we obtain
\[
x(t; s_1, s_2, \ldots, s_n; r_1, r_2, \ldots, r_n; v) = z(t - v; s_1, s_2, \ldots, s_n; r_1, r_2, \ldots, r_n)
\]
\[
= G(D)s_1 \bigg|_{t := t-v} \\
= \sum_{k=0}^{\infty} \frac{(t-v)^k}{k!} [D]^k s_1,
\]  
(32)
\[
y(t; s_1, s_2, \ldots, s_n; r_1, r_2, \ldots, r_n; v) = w(t - v; s_1, s_2, \ldots, s_n; r_1, r_2, \ldots, r_n)
\]
\[
= G(D)r_1 \bigg|_{t := t-v} \\
= \sum_{k=0}^{\infty} \frac{(t-v)^k}{k!} [D]^k r_1.
\]  
(33)

**Theorem 3.** For the system of r-ODEs with n-integer derivatives
\[
x_i^{(n)}(t; x_1, x_1', \ldots, x_1^{(n-1)}, x_2, x_2', \ldots, x_2^{(n-1)}, \ldots; x_i, x_i', \ldots, x_i^{(n-1)}),
\]
i = 1, 2, \ldots, r,
(34)

under the initial conditions
\[
x_i(t; c_{i1}, c_{i2}, \ldots, c_{i_n}; c_{i1}, c_{i2}, \ldots, c_{i_n}; \ldots; c_{(r-1)1}, c_{(r-1)2}, \ldots, c_{(r-1)n};
\]
\[c_{r1}, c_{r2}, \ldots, c_{rn}) \big|_{t=v} = c_{i1},
\]
and

\[ x_i^{(k)}(t; c_{11}, c_{12}, \ldots, c_{1n}; c_{21}, c_{22}, \ldots, c_{2n}; \ldots; c_{(r-1)1}, c_{(r-1)2}, \ldots, c_{(r-1)n}), \]

\[ c_{r1}, c_{r2}, \ldots, c_{rn} \bigg|_{t=v} = c_{i(k+1)}, \]

where \( v \in \mathbb{R} \) and \( k = 1, 2, \ldots, n - 1 \), then the solution of this system has the following operator representation:

\[ x_i(t; c_{11}, c_{12}, \ldots, c_{1n}; c_{21}, c_{22}, \ldots, c_{2n}; \ldots; c_{(r-1)1}, c_{(r-1)2}, \ldots, c_{(r-1)n});
\]

\[ c_{r1}, c_{r2}, \ldots, c_{rn}; v) = G(D)c_i \bigg|_{t=t-v}, \]

where \( D \) is the general operator of differentiation

\[ D = D_v + c_{12}D_{c_{11}} + \ldots + c_{1n}D_{c_{(n-1)}} + P_1D_{c_{1n}} + D_v + c_{22}D_{c_{21}} + \ldots + c_{2n}D_{c_{2(n-1)}} + \]

\[ + P_2D_{c_{2n}} + \ldots + D_v + c_{(r-1)2}D_{c_{(r-1)1}} + \ldots + c_{(r-1)n}D_{c_{(r-1)(n-1)}} + P_{(r-1)}D_{c_{(r-1)n}} + \]

\[ + c_{r2}D_{c_{r1}} + \ldots + c_{rn}D_{c_{r(n-1)}} + P_rD_{c_{rn}}. \]

**Proof.** It is similar to the proof of Theorem 2.

5. **Rank of Hankel Matrix**

In this section, we will try to answer on the following question:

How to determine the Hankel rank (H-rank) of a sequence \( Hr(P_j), j \in N_0 = \{0, 1, 2, \ldots\} \)? The answer will be discussed in the following steps:

1. Consider the sequence \( [P_j : j \in N_0] = \{P_0, P_1, P_2, \ldots\} \), where the elements \( P_j \) may be real, complex numbers or algebraic expressions with parameters \( s_1, s_2, \ldots \).
(2) Construct a sequence of Hankel matrices

\[
H_n = \begin{pmatrix} P_0 & P_1 & \cdots & P_{n-1} \\ P_1 & P_2 & \cdots & P_n \\ \vdots & \vdots & \ddots & \vdots \\ P_{n-1} & P_n & \cdots & P_{2n-2} \end{pmatrix}, \quad n = 1, 2, \ldots \text{, i.e., } H_1 = (P_0),
\]

\[
H_2 = \begin{pmatrix} P_0 & P_1 \\ P_1 & P_2 \end{pmatrix}, \quad H_3 = \begin{pmatrix} P_0 & P_1 & P_2 \\ P_1 & P_2 & P_3 \\ P_2 & P_3 & P_4 \end{pmatrix}\text{ and so.}
\]

(3) Construct the sequence of determinants of Hankel matrices \( \{d_1, d_2, \ldots \} \).

**Definition 6.** The sequence \([P_j : j \in N_0]\) has an H-rank \( m \in N \), \( m < \infty \), if the sequence of determinants of Hankel matrices \( d_1, d_2, \ldots, d_m, 0, 0, \ldots \), where \( d_m \neq 0 \) and \( d_{m+1} = d_{m+2} = \ldots = 0 \).

**Definition 7.** If the given sequence has an H-rank equal to \( m \), then the following characteristic equation can be constructed for that sequence [18]:

\[
\begin{vmatrix} P_0 & P_1 & \cdots & P_m \\ P_1 & P_2 & \cdots & P_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ P_{m-1} & P_m & \cdots & P_{2m-1} \\ 1 & \rho & \cdots & \rho^m \end{vmatrix} = 0, \quad (35)
\]

and the expansion of the determinant in Equation (35) yields \( m \)-th algebraic equation

\[
\rho^m + A_{m-1}\rho^{m-1} + \cdots + A_1\rho + A_0 = 0. \quad (36)
\]
**Definition 8.** If all the roots of Equation (36) are different, then [17]

\[ P_j = \sum_{r=1}^{m} \mu_r \rho_r^j, \quad (37) \]

and the coefficients \( \mu_r \) can be obtained by solving a linear system of algebraic equations (37) such that \( \rho_1, \rho_2, \ldots, \rho_m \) are determined before.

6. The Algorithm for the Construction of an Exact Solution of an Ordinary Fractional Differential Equation under the Space of Functions \( \mathcal{C} \)

In this section, we extend the algorithm introduced in ([17], [18]) to seek with our certain class of fractional differential equations. The proposed algorithm is in the following steps:

(1) Consider the following fractional differential equation with initial conditions be given:

\[ y^{(na)}(x) = P(x; y, y', \ldots, y^{(n-1)}), \quad 0 < \alpha < 1, \quad (38) \]

\[ y = y(x; s_1, s_2, \ldots, s_n), \quad y(x; s_1, s_2, \ldots, s_n)|_{x=u} = s_1, \]

\[ y^{(k)}(x; s_1, s_2, \ldots, s_n)|_{x=u} = s_{k+1}, \]

where \( k = 1, 2, \ldots, n-1. \)

(2) Using the properties of the space of functions \( \mathcal{C} \), Equation (38) can be transformed to an ordinary differential equation with integer derivatives

\[ y^{(n)}(x) = P(x; y, y', \ldots, y^{(n-1)}), \quad (39) \]

\[ y = y(x; s_1, s_2, \ldots, s_n), \quad y(x; s_1, s_2, \ldots, s_n)|_{x=u} = s_1, \]

\[ y^{(k)}(x; s_1, s_2, \ldots, s_n)|_{x=u} = s_{k+1}, \]
where \( k = 1, 2, \ldots, n - 1 \), then its solution has the following operator representation:

\[
y(x; s_1, s_2, \ldots, s_n) = \sum_{k=0}^{\infty} \frac{(x-v)^k}{k!} [D_v + s_2 D_{s_1} + \ldots + s_n D_{s_{n-1}} + P(v; s_1, s_2, \ldots, s_n) D_{s_n}]^k s_1.
\]

(3) Calculate the first elements of

\[
P_j(s_1, s_2, \ldots, s_n) = [D_v + s_2 D_{s_1} + \ldots + s_n D_{s_{n-1}} + P(v; s_1, \ldots, s_n) D_{s_n}]^k s_1,
\]

where \( P_0(s_1, s_2, \ldots, s_n) = s, \) \( P_{j+1}(s_1, s_2, \ldots, s_n) = D_{s_1} \ldots D_{s_n} P_j(s_1, s_2, \ldots, s_n) \).

The generalized operator \( D_{s_1 \ldots s_n} \) will not depend on a parameter \( v \) if

\( P(x, y, y', \ldots, y^{(n-1)}) \) does not depend on \( x \).

(4) Assume that the produced series \( P_j(s_1, s_2, \ldots, s_n) \) has an H-rank equal to \( m \), then the characteristic equation has \( m \) roots \( \rho_k(s_1, s_2, \ldots, s_n), k = 1, 2, \ldots, m \).

(5) If all roots of the characteristic equation are different, to obtain the coefficients \( \mu_r \), solve the linear system of algebraic equations corresponding to H-rank = \( m \) using Equation (37), then

\[
P_j(s_1, s_2, \ldots, s_n) = \sum_{r=1}^{m} \mu_r \rho_j^r.
\]

In other words,

\[
P_0(s_1, s_2, \ldots, s_n) = \mu_1(s_1, s_2, \ldots, s_n) + \ldots + \mu_m(s_1, s_2, \ldots, s_n),
\]

\[
P_1(s_1, s_2, \ldots, s_n) = \mu_1(s_1, s_2, \ldots, s_n) \rho_1 + \ldots + \mu_m(s_1, s_2, \ldots, s_n) \rho_m,
\]

\[
\vdots
\]

\[
P_j(s_1, s_2, \ldots, s_n) = \mu_1(s_1, s_2, \ldots, s_n) \rho_j^1 + \ldots + \mu_m(s_1, s_2, \ldots, s_n) \rho_j^m.
\]
(6) The solution of Equation (38) under the space of functions \( \mathcal{C} \) can be reformulated in the form

\[
y(x; s_1, s_2, \ldots, s_n) = \sum_{j=0}^{n} \frac{(x-v)^j}{j!} \times [\mu_1(s_1, s_2, \ldots, s_n)p_1^j(s_1, s_2, \ldots, s_n) + \ldots + \mu_m(s_1, s_2, \ldots, s_n)p_m^j(s_1, s_2, \ldots, s_n)]
\]

\[
= \sum_{r=1}^{m} \mu_r(s_1, s_2, \ldots, s_n)e^{(x-v)p_r(s_1, s_2, \ldots, s_n)}. \tag{41}
\]

(7) Assume that the produced series \( P_j(s_1, s_2, \ldots, s_n) \) has not H-rank, then we follow He and Wu method \([6]\) by using the transformation \([e^x = z, z > 0]\), then Equation (38) reads in terms of a new parameter \( Q \) such that \([Q(z) = y(\ln z)]\)

\[
Q_{zz\ldots z}^{(n)} = \frac{1}{z^n}[P_n(Q, zQ', \ldots, (az^{n-1}Q^{(n-1)}+ \ldots + bz^2Q' + czQ'))
\]

\[
+ dz^{n-1}Q^{(n-1)} + \ldots + czQ'], \tag{42}
\]

which is named as an image equation, under the initial conditions

\[
Q(e^v; s_1, s_2, \ldots, s_n) = s_1, \quad Q^{(k)}(z; s_1, s_2, \ldots, s_n)|_{z=e^v} = s_{k+1},
\]

where \( a, b, c, d, e \in \mathbb{R}; k = 1, 2, \ldots, n-1. \)

(8) The solution of Equation (42) takes the following form:

\[
Q(z; s_1, s_2, \ldots, s_n; e^v) = \sum_{j=0}^{n} \frac{(z-e^v)^j}{j!} \hat{P}_j(s_1, s_2, \ldots, s_n; e^v),
\]

where \( \hat{P}_j(s_1, s_2, \ldots, s_n; e^v) = (D_0 + s_2D_{s_1} + \ldots + s_nD_{s_{n-1}} + \hat{P}_n(v; s_1, s_2, \ldots, s_n)) \),

\[
\hat{P}_n(v; s_1, s_2, \ldots, s_n) = Q_{zz\ldots z}^{(n)}.
\]
(9) Assume that \( Hr\left( \hat{P}_n(v; s_1, \ldots, s_n) \right) = m \), and if the roots \( \rho_r(s_1, \ldots, s_n; e^v), r = 1, 2, \ldots, m \) of the characteristic equation of the sequence \( \hat{P}_j \) are different, then

\[
Q(z; s_1, s_2, \ldots, s_n; e^v) = \sum_{j=0}^{\infty} (z - e^v)^j \sum_{r=1}^{m} \mu_r(s_1, s_2, \ldots, s_n; e^v) \hat{\rho}_r^j(s_1, s_2, \ldots, s_n; e^v)
\]

\[
= \sum_{r=1}^{m} \left[ \mu_r(s_1, s_2, \ldots, s_n; e^v) \sum_{j=0}^{\infty} ((z - e^v) \hat{\rho}_r^j(s_1, s_2, \ldots, s_n; e^v))^j \right]
\]

\[
= \sum_{r=1}^{m} \frac{\mu_r(s_1, s_2, \ldots, s_n; e^v)}{1 - (z - e^v) \hat{\rho}_r^j(s_1, s_2, \ldots, s_n; e^v)}
\]

\[
= \sum_{i=1}^{n_1} o_i z^i \frac{1}{\sum_{i=1}^{n_2} p_i z^i}.
\]

(10) Inverse the variables \([z = e^x]\) gives

\[
y(x; s_1, s_2, \ldots, s_n) = \sum_{i=1}^{n_1} o_i e^{ix} \frac{1}{\sum_{i=1}^{n_2} p_i e^{ix}}, \quad (43)
\]

where \( o_i \) and \( p_i \) are real constants.

(11) If there is no Hankel rank for the sequence which resulted from the image equation, then the solution of Equation (38) can not be expressed in terms of exponential functions.

(12) If the roots of the characteristic equation are multiple, let \( Hr(p_j, j \in N_0) = m \) and the multiplicity of roots \( \rho_1, \rho_2, \ldots, \rho_l \) of the characteristic equation are \( m_1, m_2, \ldots, m_l \), then
To obtain the formula of the solution in this case, we follow the same steps (6)-(10) in Section 6.

7. Implementation of the Proposed Algorithm

Example 7.1. In this example, we consider a linear system of fractional differential equations and under the space of functions \( \mathcal{C} \), we want to simplify the existence or non-existence of a solution for it in terms of exp-functions and obtain this solution

\[
\frac{1}{2} D^\frac{1}{2} y_1 = y_1 + y_2, \quad (44)
\]

\[
\frac{1}{2} D^\frac{1}{2} y_2 = -y_1 + y_2, \quad (45)
\]

subject to the initial conditions

\[
y_1(x)|_{x=v} = s \quad \text{and} \quad y_2(x)|_{x=v} = t.
\]

This system can be reduced to

\[
y_1' = \frac{1}{2} y_2, \quad (46)
\]

\[
y_2' = \frac{1}{2} y_1. \quad (47)
\]

The algorithm of the solution in terms of exp-functions

(1) Under the space of functions \( \mathcal{C} \), we reduce the system of fractional differential equations (44)-(45) to the system of ordinary differential equations with integer derivatives (46)-(47).

(2) Using the proposed operator method

\[
y_1(x; s, t) = \sum_{j=0}^{\infty} \frac{(x-v)^j}{j!} U_j(s, t) = \sum_{j=0}^{\infty} \frac{(x-v)^j}{j!} \left[ D_v + P(v; s, t) D_s + Q(v; s, t) D_j \right]^j s,
\]

(48)
\[ y_2(x; s, t) = \sum_{j=0}^{\infty} \frac{(x - v)^j}{j!} V_j(s, t) = \sum_{j=0}^{\infty} \frac{(x - v)^j}{j!} [D_0 + P(v; s, t)D_0 + Q(v; s, t)D_0^j]t. \]

(49)

To determine the possibility of the existence of a solution in terms of exponential functions for the first component of the vector solution, \( y_1 \), we follow the following steps:

(3) Consider the sequence \( U_j(s, t), j \in N_0 = \{0, 1, 2, \ldots\} \) such that

\[ U_j(s, t) = \left[ \frac{t}{2} \ D_0 + \frac{s}{2} \ D_0 \right]^j s, \]

then \( U_0 = s, U_1 = \frac{t}{2}, U_2 = \frac{s}{4}, U_3 = \frac{t}{8}, U_4 = \frac{s}{16} \), and so.

(4) Construct the sequence of Hankel matrices on the form

\[
H_1 = (U_0) = (s), \quad H_2 = \begin{pmatrix} U_0 & U_1 \\ U_1 & U_2 \end{pmatrix} = \begin{pmatrix} s & \frac{t}{2} \\ \frac{t}{2} & \frac{s}{4} \end{pmatrix},
\]

\[
H_3 = \begin{pmatrix} U_0 & U_1 & U_2 \\ U_1 & U_2 & U_3 \\ U_2 & U_3 & U_4 \end{pmatrix} = \begin{pmatrix} s & \frac{t}{2} & \frac{s}{4} \\ \frac{t}{2} & \frac{s}{4} & \frac{t}{8} \\ \frac{s}{4} & \frac{t}{8} & \frac{s}{16} \end{pmatrix}, \text{ and so.}
\]

(5) Construct the sequence of determinants of Hankel matrices on the form

\[
d_1 = s, \quad d_2 = \frac{s^2}{4} - \frac{t^2}{4}, \quad d_3 = d_4 = \ldots = 0.
\]

(6) Simplify the Hankel rank of \( U_j(s, t) \);

\[
Hr[U_j(s, t), j = 0, 1, 2, \ldots] = 2 \Leftrightarrow d_2 \neq 0.
\]
(7) Construct the characteristic equation of the sequence $U_j(s, t)$ on the form

\[
\begin{vmatrix}
U_0 & U_1 & U_2 \\
U_1 & U_2 & U_3 \\
1 & \rho & \rho^2
\end{vmatrix} = \begin{vmatrix} s & \frac{t}{2} & \frac{s}{4} \\
\frac{t}{2} & \frac{s}{4} & \frac{t}{8} \\
1 & \rho & \rho^2
\end{vmatrix} = 0,
\]

then expanding the determinant produces

\[
\frac{s^2}{16} + \frac{t^2}{16} + \frac{s^2 \rho^2}{4} - \frac{t^2 \rho^2}{4} = 0, \tag{50}
\]

solve the algebraic equation (50), we obtain

\[
\rho_1 = \frac{1}{2} \quad \text{and} \quad \rho_2 = \frac{-1}{2}.
\]

(8) Solve the linear system of algebraic equations

\[
U_j(s, t) = \sum_{r=1}^{2} \mu_r(s, t)\rho_r^j = \mu_1(s, t)\rho_1^j + \mu_2(s, t)\rho_2^j, \quad j \in \mathbb{N}_0 = \{0, 1, 2, \ldots\},
\]

for $j = 0$, we obtain $U_0(s, t) = \mu_1(s, t) + \mu_2(s, t) = s$; for $j = 1$, we obtain $U_1(s, t) = \mu_1(s, t)\rho_1 + \mu_2(s, t)\rho_2 = \frac{t}{2}$; solve these two algebraic equations to obtain

\[
\mu_1(s, t) = \frac{t + s}{2} \quad \text{and} \quad \mu_2(s, t) = \frac{s - t}{2}.
\]

(9) The elements of the sequence can be written in the following general form:

\[
U_j(s, t) = \frac{t + s}{2} \rho_1^j + \frac{s - t}{2} \rho_2^j. \tag{52}
\]
(10) Under the space of functions $\mathcal{C}$, the first component of the vector solution, $y_1$, of the linear system using operator method can be written in the form

$$y_1(x; s, t) = \sum_{j=0}^{\infty} U_j(s, t) \frac{e^{-v} v^j}{j!},$$

$$= U_0(s, t) + \frac{(x-v)}{1!} U_1(s, t) + \frac{(x-v)^2}{2!} U_2(s, t) + \ldots$$

$$= \sum_{r=1}^{2} \mu_r(s, t) \rho_r^0 + \sum_{r=1}^{2} \mu_r(s, t) \frac{(x-v)}{1!} \rho_r^1 + \sum_{r=1}^{2} \mu_r(s, t) \frac{(x-v)^2}{2!} \rho_r^2 + \ldots$$

$$= \sum_{r=1}^{2} \mu_r(s, t) \left[ \sum_{j=0}^{\infty} \frac{(\rho_r(x-v))^j}{j!} \right] = \sum_{r=1}^{2} \mu_r(s, t) e^{(x-v)\rho_r}$$

$$= \left[ \frac{t+s}{2} \right] e^{-\frac{(x-v)}{2}} + \left[ \frac{s-t}{2} \right] e^{\frac{(x-v)}{2}}. \quad (53)$$

So, we can express the first component of the vector solution, $y_1$, of the linear system inside the space of functions $\mathcal{C}$ as a finite sum of exponential functions.

To determine the possibility of the existence of a solution in terms of exponential functions for the second component of the vector solution, $y_2$, we repeat the above steps from (3)-(10) on the form:

(1) Consider the sequence $V_j(s, t), j \in N_0 = \{0, 1, 2, \ldots \}$ such that $V_j(s, t) = \left[ \frac{t}{2} D_s + \frac{s}{2} D_t \right]^j t$, then $V_0 = t, V_1 = \frac{s}{2}, V_2 = \frac{t}{4}, V_3 = \frac{s}{8}$, and $V_4 = \frac{t}{16}$, and so.

(2) Construct the sequence of Hankel matrices on the form

$$H_1 = (V_0) = (t), \quad H_2 = \begin{pmatrix} V_0 & V_1 \\ V_1 & V_2 \end{pmatrix} = \begin{pmatrix} t & \frac{s}{2} \\ \frac{s}{2} & \frac{t}{4} \end{pmatrix},$$
(3) Construct the sequence of determinants of Hankel matrices on the form

\[ d_1 = t, \quad d_2 = -\frac{s^2}{4} + \frac{t^2}{4}, \quad d_3 = d_4 = \ldots = 0. \]

(4) Simplify the Hankel rank of \( V_j(s, t) \)

\[ Hr[V_j(s, t), j = 0, 1, 2, \ldots] = 2 \Leftrightarrow d_2 \neq 0. \]

(5) Construct the characteristic equation of the sequence \( V_j(s, t) \) on the form

\[
\begin{vmatrix}
V_0 & V_1 & V_2 & t & \frac{s}{2} & \frac{t}{4} \\
V_1 & V_2 & V_3 & \frac{s}{2} & \frac{t}{4} & \frac{s}{8} \\
1 & \rho & \rho^2 & 1 & \rho & \rho^2
\end{vmatrix} = 0,
\]

then expanding the determinant produces

\[
\frac{s^2}{16} - \frac{t^2}{16} - \frac{s^2 \rho^2}{4} + \frac{t^2 \rho^2}{4} = 0, \quad (54)
\]

solve the algebraic equation (54), we obtain

\[
\rho_1 = \frac{1}{2} \quad \text{and} \quad \rho_2 = -\frac{1}{2}.
\]

(6) Solve the linear system of algebraic equations

\[
V_j(s, t) = \sum_{r=1}^{2} \mu_r(s, t)\rho^j_r = \mu_1(s, t)\rho^j_1 + \mu_2(s, t)\rho^j_2, \quad (55)
\]
for \( j = 0 \), we obtain \( V_0(s, t) = \mu_1(s, t) + \mu_2(s, t) = t; \) for \( j = 1 \), we obtain \( V_1(s, t) = \mu_1(s, t)\rho_1 + \mu_2(s, t)\rho_2 = \frac{s}{2} \); solve these two algebraic equations to obtain

\[
\mu_1(s, t) = \frac{t + s}{2} \quad \text{and} \quad \mu_2(s, t) = \frac{t - s}{2}.
\]

(7) The elements of the sequence can be written in the following general form:

\[
V_j(s, t) = \frac{t + s}{2} \rho_1^j + \frac{t - s}{2} \rho_2^j. \tag{56}
\]

(8) Under the space of functions \( \mathcal{C} \), the second component of the vector solution, \( y_2 \), of the linear system using operator method can be written in the form

\[
y_2(x; s, t) = \sum_{j=0}^{\infty} V_j(s, t) \left(\frac{x - v}{j!}\right)^j = V_0(s, t) + \left(\frac{x - v}{1!}\right) V_1(s, t) + \left(\frac{x - v}{2!}\right)^2 V_2(s, t) + \ldots
\]

\[
= \sum_{r=1}^{2} \mu_r(s, t) \rho_r^0 + \sum_{r=1}^{2} \mu_r(s, t) \left(\frac{x - v}{1!}\right) \rho_r^1 + \sum_{r=1}^{2} \mu_r(s, t) \left(\frac{x - v}{2!}\right)^2 \rho_r^2 + \ldots
\]

\[
= \sum_{r=1}^{2} \mu_r(s, t) \left[ \sum_{j=0}^{\infty} \left(\rho_r(x - v)\right)^j \right] = \sum_{r=1}^{2} \mu_r(s, t) e^{(x-v)\rho_r}
\]

\[
= \left[ \frac{t + s}{2} \right] e^{\frac{x-v}{2}} + \left[ \frac{t - s}{2} \right] e^{-\frac{x-v}{2}}. \tag{57}
\]

So, we can express the second component of the vector solution, \( y_2 \), of the linear system inside the space of functions \( \mathcal{C} \) as a finite sum of exponential functions.
Example 7.2. Consider a linear system of fractional differential equations and under the space of functions $\mathcal{C}$, we want to simplify the existence or non-existence of a solution for it in terms of exp-functions and obtain this solution

\[
\frac{1}{2} D^3 y_1 = y_1 + 2y_2, \quad (58)
\]

\[
\frac{1}{2} D^3 y_2 = -y_1 + y_2, \quad (59)
\]

subject to the initial conditions

\[
y_1(x)|_{x=0} = s \quad \text{and} \quad y_2(x)|_{x=0} = t.
\]

This system is reduced to the following form by using the properties of class $\mathcal{C}$

\[
y_1' = -\frac{81}{17} y_1 + \frac{54}{17} y_2, \quad (60)
\]

\[
y_2' = -\frac{9}{17} y_1 + \frac{45}{17} y_2. \quad (61)
\]

The algorithm of the solution in terms of exp-functions

1. Under the space of functions $\mathcal{C}$ reduce the system of fractional differential equations (58)-(59) to the system of ordinary differential equations with integer derivatives (60)-(61).

2. Using the proposed operator method

\[
y_1(x; s, t) = \sum_{j=0}^{\infty} \frac{(x - s)^j}{j!} U_j(s, t) = \sum_{j=0}^{\infty} \frac{(x - s)^j}{j!} [D_0 + P(v; s, t)D_k + Q(v; s, t)D_{l}] s,
\]

\[
y_2(x; s, t) = \sum_{j=0}^{\infty} \frac{(x - s)^j}{j!} V_j(s, t) = \sum_{j=0}^{\infty} \frac{(x - s)^j}{j!} [D_0 + P(v; s, t)D_k + Q(v; s, t)D_{l}] t.
\]

\[
(62)
\]

\[
(63)
\]
To determine the possibility of the existence of a solution in terms of exponential functions for the first component of the vector solution, $y_1$, we follow the following steps:

(3) Consider the sequence $U_j(s, t), j \in N_0 = \{0, 1, 2, \ldots\}$ such that

$$U_j(s, t) = \left[ ( -\frac{81}{17} s + \frac{54}{17} t ) D_s + ( -\frac{9}{17} s + \frac{45}{17} t ) D_t \right]^j.$$ 

then $U_0 = s$, $U_1 = ( -\frac{81}{17} s + \frac{54}{17} t )$, $U_2 = \frac{54}{17} ( -\frac{9}{17} s + \frac{45}{17} t ) - \frac{81}{17} ( -\frac{81}{17} s + \frac{54}{17} t )$, $U_3 = -\frac{1944}{289} ( -\frac{9}{17} s + \frac{45}{17} t ) + \frac{6075}{289} ( -\frac{81}{17} s + \frac{54}{17} t )$, $U_4 = \frac{240570}{4913} ( -\frac{9}{17} s + \frac{45}{17} t ) - \frac{474579}{4913} ( -\frac{81}{17} s + \frac{54}{17} t )$, and so.

(4) Construct the sequence of Hankel matrices on the form

$$H_1 = (U_0) = (s), \quad H_2 = \begin{pmatrix} U_0 & U_1 \\ U_1 & U_2 \end{pmatrix}, \quad H_3 = \begin{pmatrix} U_0 & U_1 & U_2 \\ U_1 & U_2 & U_3 \\ U_2 & U_3 & U_4 \end{pmatrix}, \quad \text{and so.}$$

(5) Construct the sequence of determinants of Hankel matrices on the form

$$d_1 = s, \quad d_2 = \frac{1}{289} ( -486 s^2 + 6804 st - 2916 t^2 ), \quad d_3 = d_4 = \ldots = 0.$$

(6) Simplify the Hankel rank of $U_j(s, t)$

$$Hr[U_j(s, t), j = 0, 1, 2, \ldots] = 2 \Leftrightarrow d_2 \neq 0.$$
(7) Construct the characteristic equation of the sequence $U_j(s, t)$ on the form

$$
\begin{vmatrix}
  s & (-\frac{81}{17} s + \frac{54}{17} t) \\
  (-\frac{81}{17} s + \frac{54}{17} t) & 1 \\
  \frac{54}{17} (-\frac{17}{9 s} + \frac{45}{17} t) - \frac{81}{17} (-\frac{81 s}{17} + \frac{54 t}{17}) & \rho
\end{vmatrix} = 0,
$$

then expanding the determinant produces

$$
\frac{153574}{83521} s^2 - \frac{21493836}{83521} st + \frac{9211644}{83521} t^2 - \frac{17496}{4913} s^2 \rho^2 + \frac{244944}{4913} st \rho \\
+ \frac{6804}{289} s^2 \rho^2 - \frac{104976}{4913} t^2 \rho - \frac{486}{289} s^2 \rho^2 - \frac{2916}{289} t^2 \rho^2 = 0, \quad (64)
$$

solve the algebraic equation (64), we obtain

$$
\rho_1 = \frac{9}{17}(-2 - \sqrt{43}) \quad \text{and} \quad \rho_2 = \frac{9}{17}(-2 + \sqrt{43}).
$$

(8) Solve the linear system of algebraic equations

$$
U_j(s, t) = \sum_{r=1}^{2} \mu_r(s, t) \rho_r^j = \mu_1(s, t) \rho_1^j + \mu_2(s, t) \rho_2^j, \quad (65)
$$

for $j = 0$, we obtain $U_0(s, t) = \mu_1(s, t) + \mu_2(s, t) = s$; for $j = 1$, we obtain $U_1(s, t) = \mu_1(s, t) \rho_1 + \mu_2(s, t) \rho_2 = (-\frac{81}{17} s + \frac{54}{17} t)$; solve these two algebraic equations to obtain

$$
\mu_1(s, t) = \frac{7s + \sqrt{43}s - 6t}{2\sqrt{43}} \quad \text{and} \quad \mu_2(s, t) = \frac{43s - 7\sqrt{43}s + 7\sqrt{43}t}{86}.
$$
(9) The elements of the sequence can be written in the following form:

\[
U_j(s, t) = \left[ \frac{7s + \sqrt{43}s - 6t}{2\sqrt{43}} \right] \rho_1^j + \left[ \frac{43s - 7\sqrt{43}s + 7\sqrt{43}t}{86} \right] \rho_2^j.
\]  

(66)

(10) Under the space of functions \( \mathcal{C} \), the first component of the vector solution, \( y_1 \), of the linear system using operator method can be written in the form

\[
y_1(x; s, t) = \sum_{j=0}^{\infty} U_j(s, t) \frac{(x-v)^j}{j!}
= U_0(s, t) + \frac{(x-v)}{1!} U_1(s, t) + \frac{(x-v)^2}{2!} U_2(s, t) + \ldots
= \sum_{r=1}^{2} \mu_r(s, t) \rho_0^r + \sum_{r=1}^{2} \mu_r(s, t) \frac{(x-v)^1}{1!} \rho_1^r + \sum_{r=1}^{2} \mu_r(s, t) \frac{(x-v)^2}{2!} \rho_2^r + \ldots
= \sum_{r=1}^{2} \mu_r(s, t) \left[ \sum_{j=0}^{\infty} \frac{(x-v)^j}{j!} \right] = \sum_{r=1}^{2} \mu_r(s, t) e^{(x-v)\rho_r}
\]

\[
= \left[ \frac{7s + \sqrt{43}s - 6t}{2\sqrt{43}} \right] \frac{9}{17} e^{(x-v)(-2+\sqrt{43})} + \left[ \frac{43s - 7\sqrt{43}s + 7\sqrt{43}t}{86} \right] \frac{9}{4} e^{(x-v)(-2+\sqrt{43})}.
\]  

(67)

So, we can express the first component of the vector solution, \( y_1 \), of the linear system inside the space of functions \( \mathcal{C} \) as a finite sum of exponential functions.

To determine the possibility of the existence of a solution in terms of exponential functions for the second component of the vector solution, \( y_2 \), we repeat the above steps from (3)-(10).

(1) Consider the sequence \( V_j(s, t), j \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \) such that

\[
V_j(s, t) = \left[ \left( \frac{-81}{17}s + \frac{54}{17}t \right) D_s + \left( -\frac{9}{17}s + \frac{45}{17}t \right) D_t \right] J^j t,
\]
then $V_0 = t$, $V_1 = \left(\frac{-9}{17} s + \frac{45}{17} t\right)$, $V_2 = \frac{45}{17} \left(\frac{-9}{17} s + \frac{45}{17} t\right) - \frac{9}{17} \left(\frac{-81}{17} s + \frac{54}{17} t\right)$,

$$V_3 = \frac{1539}{289} \left(\frac{-9}{17} s + \frac{45}{17} t\right) + \frac{324}{289} \left(\frac{-81}{17} s + \frac{54}{17} t\right), \quad V_4 = \frac{5103}{4913} \left(\frac{-9}{17} s + \frac{45}{17} t\right) - \frac{40095}{4913} \left(\frac{-81}{17} s + \frac{54}{17} t\right),$$

and so.

(2) Construct the sequence of Hankel matrices on the form

$$H_1 = (V_0) = (t), \quad H_2 = \left(\begin{array}{ccc} V_0 & V_1 \\ V_1 & V_2 \end{array}\right), \quad H_3 = \left(\begin{array}{ccc} V_0 & V_1 & V_2 \\ V_1 & V_2 & V_3 \end{array}\right), \quad H_4 = \left(\begin{array}{cccc} V_0 & V_1 & V_2 & V_3 \\ V_1 & V_2 & V_3 & V_4 \end{array}\right).$$

(3) Construct the sequence of determinants of Hankel matrices on the form

$$d_1 = t, \quad d_2 = \frac{1}{289} \left(-81s^2 + 1134st - 486t^2\right), \quad d_3 = d_4 = \ldots = 0.$$

(4) Simplify the Hankel rank of $V_j(s, t)$

$$Hr[V_j(s, t), \quad j = 0, 1, 2, \ldots] = 2 \Leftrightarrow d_2 \neq 0.$$

(5) Construct the characteristic equation of the sequence $V_j(s, t)$ on the form

$$\left|\begin{array}{ccc} t & (\frac{-9}{17} s + \frac{45}{17} t) \\ (\frac{-9}{17} s + \frac{45}{17} t) & \frac{45}{17} \left(\frac{-9}{17} s + \frac{45}{17} t\right) - \frac{9}{17} \left(\frac{-81}{17} s + \frac{54}{17} t\right) \\ \frac{1539}{289} \left(\frac{-9}{17} s + \frac{45}{17} t\right) + \frac{324}{289} \left(\frac{-81}{17} s + \frac{54}{17} t\right) & \frac{5103}{4913} \left(\frac{-9}{17} s + \frac{45}{17} t\right) - \frac{40095}{4913} \left(\frac{-81}{17} s + \frac{54}{17} t\right) \end{array}\right| = 0,$$

then expanding the determinant produces
solving the algebraic equation (68), we obtain

$$\rho_1 = \frac{9}{17} (-2 - \sqrt{43}) \quad \text{and} \quad \rho_2 = \frac{9}{17} (-2 + \sqrt{43}).$$

(6) Solve the linear system of algebraic equations

$$V_j(s, t) = \sum_{r=1}^{2} \mu_r(s, t) \rho_r^j = \mu_1(s, t) \rho_1^j + \mu_2(s, t) \rho_2^j,$$  \hspace{1cm} (69)

for \( j = 0 \), we obtain \( V_0(s, t) = \mu_1(s, t) + \mu_2(s, t) = t \); for \( j = 1 \), we obtain

$$V_1(s, t) = \mu_1(s, t) \rho_1 + \mu_2(s, t) \rho_2 = \left( -\frac{9}{17} s + \frac{45}{17} t \right); \text{ solve these two algebraic equations to obtain}$$

$$\mu_1(s, t) = \frac{s + \sqrt{43} t - 7 t}{2 \sqrt{43}} \quad \text{and} \quad \mu_2(s, t) = \frac{43 t - \sqrt{43} s + 7 43 t}{86}.$$

(7) The elements of the sequence can be written in the following form:

$$V_j(s, t) = \frac{s + \sqrt{43} t - 7 t}{2 \sqrt{43}} \rho_1^j + \frac{43 t - \sqrt{43} s + 7 \sqrt{43} t}{86} \rho_2^j.$$  \hspace{1cm} (70)

(8) Under the space of functions \( \zeta \), the second component of the vector solution, \( y_2 \), of the linear system using operator method can be written in the form

$$y_2(x; s, t) = \sum_{j=0}^{\infty} V_j(s, t) \frac{(x-u)^j}{j!}$$

$$= V_0(s, t) + \frac{(x-u)}{1!} V_1(s, t) + \frac{(x-u)^2}{2!} V_2(s, t) + ...$$
\[
\begin{aligned}
&= \sum_{r=1}^{2} \mu_r(s, t) \rho_r^0 + \sum_{r=1}^{2} \mu_r(s, t) \frac{(x - u)}{1!} \rho_r^1 + \sum_{r=1}^{2} \mu_r(s, t) \frac{(x - u)^2}{2!} \rho_r^2 + \\
&= \sum_{r=1}^{2} \mu_r(s, t) \left[ \sum_{j=0}^{\infty} \frac{(x - u) \rho_r^j}{j!} \right] = \sum_{r=1}^{2} \mu_r(s, t) e^{(x-u) \rho_r} \\
&= \left[ s + \frac{\sqrt{433} t - 7t}{2\sqrt{43}} \right] \frac{9}{e^{11t}}(x - u)(-2 - \sqrt{435}) + \left[ \frac{43t - \sqrt{433} + 7\sqrt{43}t}{86} \right] \frac{9}{e^{11t}}(x - u)(-2 + \sqrt{43}) \\
\end{aligned}
\]

(71)

So, we can express the second component of the vector solution, \( y_2 \), of this linear system inside the space of functions \( \mathcal{C} \) as a finite sum of exponential functions.

**Example 7.3.** Consider non-linear system of fractional differential equations and under the space of functions \( \mathcal{C} \), we want to simplify the existence or non-existence of a solution for it in terms of exp-functions and obtain this solution if it exists

\[
\frac{1}{D^2} y_1 = -y_1, \quad (72)
\]

\[
\frac{1}{D^2} y_2 = y_1 - y_2^2, \quad (73)
\]

\[
\frac{1}{D^2} y_3 = y_2^2, \quad (74)
\]

subject to the initial conditions

\[ y_1(x)\big|_{x=u} = s_1, \quad y_2(x)\big|_{x=u} = s_2, \quad \text{and} \quad y_3(x)\big|_{x=u} = s_3. \]

This system can be reduced to

\[
y_1' = y_1, \quad (75)
\]

\[
y_2' = -y_1 - (y_1 - y_2^2)(y_2 + \frac{3}{8}), \quad (76)
\]
\[ y'_3 = (y_1 - y_2^2)(y_2 + \frac{3}{8}). \]  

(77)

**The algorithm of the solution in terms of exp-functions**

(1) Under the space of functions \( \mathcal{C} \) reduce the system of fractional differential equations (72)-(74) to the system of ordinary differential equations with integer derivatives (75)-(77).

(2) Using the proposed operator method

\[
y_1(x; s_1, s_2, s_3) = \sum_{j=0}^{\infty} \frac{(x - v)^j}{j!} U_j(s_1, s_2, s_3),
\]

(78)

\[
y_2(x; s_1, s_2, s_3) = \sum_{j=0}^{\infty} \frac{(x - v)^j}{j!} V_j(s_1, s_2, s_3),
\]

(79)

\[
y_3(x; s_1, s_2, s_3) = \sum_{j=0}^{\infty} \frac{(x - v)^j}{j!} W_j(s_1, s_2, s_3),
\]

(80)

where

\[
U_j(s_1, s_2, s_3) = [D_v + P_1(v; s_1, s_2, s_3)D_{s_1} + P_2(v; s_1, s_2, s_3)D_{s_2} + P_3(v; s_1, s_2, s_3)D_{s_3})^j s_1,
\]

\[
V_j(s_1, s_2, s_3) = [D_v + P_1(v; s_1, s_2, s_3)D_{s_1} + P_2(v; s_1, s_2, s_3)D_{s_2} + P_3(v; s_1, s_2, s_3)D_{s_3})^j s_2,
\]

\[
W_j(s_1, s_2, s_3) = [D_v + P_1(v; s_1, s_2, s_3)D_{s_1} + P_2(v; s_1, s_2, s_3)D_{s_2} + P_3(v; s_1, s_2, s_3)D_{s_3})^j s_3.
\]

To determine the possibility of the existence of a solution in terms of exponential functions for the first component of the vector solution, \( y_1 \), we follow the following steps:
(3) Consider the sequence \( U_j(s_1, s_2, s_3), j \in N_0 = \{0, 1, 2, \ldots\} \) such that
\[
U_j(s_1, s_2, s_3) = [s_1 D_{s_1} + (-s_1 - (s_1 - s_2^2)(s_2 + \frac{3}{8})) D_{s_2} + ((s_1 - s_2^2)(s_2 + \frac{3}{8})) D_{s_3}]^j s_1,
\]
then \( U_0 = s_1, U_1 = s_1, U_2 = s_1, U_3 = s_1, U_4 = s_1 \), and so.

(4) Construct the sequence of Hankel matrices on the form
\[
H_1 = (s_1), \quad H_2 = \begin{pmatrix} s_1 & s_1 \\ s_1 & s_1 \end{pmatrix}, \quad \text{and so.}
\]

(5) Construct the sequence of determinants of Hankel matrices.

(6) Simplify the Hankel rank of \( U_j(s_1, s_2, s_3), j \in \{0, 1, 2, \ldots\} \) \( \Rightarrow d_1 \neq 0 \).

(7) Construct the characteristic equation of the sequence
\[
U_j(s_1, s_2, s_3) \] on the form
\[
\begin{vmatrix} s_1 & s_1 \\ 1 & \rho \end{vmatrix} = 0,
\]
then expanding the determinant produces
\[
s_1 \rho - s_1 = 0, \quad (81)
\]
solve the algebraic equation (81), we obtain \( \rho = 1 \).

(8) Solve the linear system of algebraic equations
\[
U_j(s_1, s_2, s_3) = \mu_1(s_1, s_2, s_3) \rho^j, \quad (82)
\]
for \( j = 0 \), we obtain \( U_0(s_1, s_2, s_3) = \mu_1(s_1, s_2, s_3) = s_1 \).

(9) The elements of the sequence can be written in the following general form:
\[
U_j(s_1, s_2, s_3) = s_1 \rho^j = s_1. \quad (83)
\]
(10) Under the space of functions $\mathcal{C}$, the first component of the vector solution, $y_1$, of the non-linear system using operator method can be written in the form

$$y_1(x; s_1, s_2, s_3) = \sum_{j=0}^{\infty} U_j(s_1, s_2, s_3) \frac{(x - v)^j}{j!}$$

$$= U_0(s_1, s_2, s_3) + \frac{(x - v)}{1!} U_1(s_1, s_2, s_3) + \frac{(x - v)^2}{2!} U_2(s_1, s_2, s_3) + \ldots$$

$$= \mu_1(s_1, s_2, s_3) + \frac{(x - v)}{1!} \mu_1(s_1, s_2, s_3) + \frac{(x - v)^2}{2!} \mu_2 + \ldots$$

$$= \mu_1(s_1, s_2, s_3) \left[ \sum_{j=0}^{\infty} \rho \frac{(x - v)^j}{j!} \right]$$

$$= se^{(x - v)}. \hspace{1cm} (84)$$

So, we can express the first component of the vector solution, $y_1$, of this non-linear system inside the space of functions $\mathcal{C}$ as a finite sum of exponential functions.

To determine the possibility of the existence of a solution in terms of exponential functions for the second component of the vector solution, $y_2$, we follow the following steps:

(1) Consider the sequence $V_j(s_1, s_2, s_3)$, $j \in N_0 = \{0, 1, 2, \ldots\}$ such that

$$V_j(s_1, s_2, s_3) = [s_1 D_{s_1} + (-s_1 - (s_1 - s_2^2)(s_2 + \frac{3}{8})) D_{s_2} + ((s_1 - s_2^2)(s_2 + \frac{3}{8})) D_{s_3}]^j s_2,$$

then $V_0 = s_2$, $V_1 = -s_1 - (s_1 - s_2^2)(s_2 + \frac{3}{8})$. 
$$V_2 = s_1(-\frac{11}{8} - s_2) + (-s_1 + s_2^2 + 2s_2(\frac{3}{8} + s_2))(-s_1 - (\frac{3}{8} + s_2)(s_1 - s_2^2)),$$

$$V_3 = s_1(-\frac{11}{8}) + s_1 - s_2 + (\frac{3}{8} + s_2)(s_1 - s_2^2) + (-\frac{11}{8}) - s_2(-s_1 + s_2^2 + 2s_2(\frac{3}{8} + s_2)) + (-s_1 - (\frac{3}{8} + s_2)(s_1 - s_2^2))(-s_1 + (-s_1 + s_2^2 + 2s_2(\frac{3}{8} + s_2))^2 + (4s_2 + 2(\frac{3}{8} + s_2))(-s_1 - (\frac{3}{8} + s_2)(s_1 - s_2^2)), \text{ and so.}$$

2. Construct the sequence of Hankel matrices.

3. Construct the sequence of determinants of Hankel matrices on the form

$$d_1 = s_2, \quad d_2 = -\left(\frac{121s_1^2}{64}\right) - \left(\frac{11s_1s_2}{8}\right) - \left(\frac{11s_1^2s_2}{8}\right) - s_1s_2^2 - \left(\frac{7s_1s_2^3}{4}\right) + \left(\frac{9s_2^4}{64}\right) - 2s_1s_2^4 + \left(\frac{9s_2^5}{8}\right) + 2s_2^6,$$

$$d_3, d_4, ... \neq 0, \text{ then } V_j(s_1, s_2, s_3) \text{ has no Hankel rank.}$$

4. Introduce a new parameter such that $e^x = z$, then $\omega = \omega(x) = y(\ln(x))$, and rewrite the system (75)-(77) corresponding to this new parameter

$$\omega'_1 = \frac{1}{z}\omega_1, \quad (85)$$

$$\omega'_2 = \frac{1}{z}(-\omega_1 - (\omega_1 - \omega_2^2)(\omega_2 + \frac{3}{8})), \quad (86)$$

$$\omega'_3 = \frac{1}{z}(\omega_1 - \omega_2^2)(\omega_2 + \frac{3}{8}), \quad (87)$$

subject to the initial conditions

$$\omega_1(z)|_{z=e^v} = s_1, \quad \omega_2(z)|_{z=e^v} = s_2, \quad \text{and} \quad \omega_3(z)|_{z=e^v} = s_3,$$

then the solution of the image equation will have the form
\[ \omega_2(z; s_1, s_2, s_3) = \sum_{j=0}^{\infty} \frac{(z-v)^j}{j!} \hat{V}_j(s_1, s_2, s_3). \] (88)

(5) The coefficients of this image equation

\[ \hat{V}_j(s_1, s_2, s_3) = \frac{1}{j!} \left[ \frac{1}{v} (s_1 D_{s_1} + (s_1 - (s_1 - s_2^2)(s_2 + \frac{3}{8})D_{s_2} + ((s_1 - s_2^2)(s_2 + \frac{3}{8})D_{s_3} \right)^j s_2, \]

\[ j \in \{0, 1, 2, \ldots\} \] have no Hankel rank \( \text{Hr} \left( \hat{V}_j(s_1, s_2, s_3) \right) \),

\[ j \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \] is unknown.

So, the second component of the vector solution, \( y_2 \), cannot be expressed through finite sum or ratio of exponential functions also.

To determine the possibility of the existence of a solution in terms of exponential functions for the third component of the vector solution, \( y_3 \), we follow the following steps:

(1) Consider the sequence \( W_j(s_1, s_2, s_3) \), \( j \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \) such that

\[ W_j(s_1, s_2, s_3) = [s_1 D_{s_1} + (s_1 - (s_1 - s_2^2)(s_2 + \frac{3}{8})D_{s_2} + ((s_1 - s_2^2)(s_2 + \frac{3}{8})D_{s_3} \right]^j s_3, \]

then

\[ W_0 = s_3, \quad W_1 = (s_1 - s_2^2)(s_2 + \frac{3}{8}), \]

\[ W_2 = s_1 (\frac{3}{8} + s_2) + (s_1 - s_2^2 - 2s_2 (\frac{3}{8} + s_2)) (s_1 - (s_1 - s_2^2)(s_2 + \frac{3}{8})D_{s_2} + ((s_1 - s_2^2)(s_2 + \frac{3}{8})D_{s_3} \right)^j s_3, \]

\[ W_3 = s_1 (\frac{3}{8} - s_1 + s_2 - (\frac{3}{8} + s_2)(s_1 - s_2^2)) + (s_1 - (s_1 - s_2^2)(s_2 + \frac{3}{8})D_{s_2} + ((s_1 - s_2^2)(s_2 + \frac{3}{8})D_{s_3} \right)^j s_3, \]

\[ W_4 = s_1 (\frac{3}{8} - s_1 + s_2 - (\frac{3}{8} + s_2)(s_1 - s_2^2)) + (s_1 - (s_1 - s_2^2)(s_2 + \frac{3}{8})D_{s_2} + ((s_1 - s_2^2)(s_2 + \frac{3}{8})D_{s_3} \right)^j s_3, \]

\[ W_5 = s_1 (\frac{3}{8} - s_1 + s_2 - (\frac{3}{8} + s_2)(s_1 - s_2^2)) + (s_1 - (s_1 - s_2^2)(s_2 + \frac{3}{8})D_{s_2} + ((s_1 - s_2^2)(s_2 + \frac{3}{8})D_{s_3} \right)^j s_3, \]

\[ W_6 = s_1 (\frac{3}{8} - s_1 + s_2 - (\frac{3}{8} + s_2)(s_1 - s_2^2)) + (s_1 - (s_1 - s_2^2)(s_2 + \frac{3}{8})D_{s_2} + ((s_1 - s_2^2)(s_2 + \frac{3}{8})D_{s_3} \right)^j s_3, \]

\[ W_7 = s_1 (\frac{3}{8} - s_1 + s_2 - (\frac{3}{8} + s_2)(s_1 - s_2^2)) + (s_1 - (s_1 - s_2^2)(s_2 + \frac{3}{8})D_{s_2} + ((s_1 - s_2^2)(s_2 + \frac{3}{8})D_{s_3} \right)^j s_3, \]
\[ + (-4s_2 - 2(\frac{3}{8} + s_2))(-s_1 - (\frac{3}{8} + s_2)(s_1 - s_2^2)), \text{ and so.} \]

(2) Construct the sequence of Hankel matrices.

(3) Construct the sequence of determinants of Hankel matrices on the form \( d_1 = s_3, \) and \( d_2, d_3, d_4, \ldots \neq 0, \) then, \( W_j(s_1, s_2, s_3) \) has no Hankel rank.

(4) Introduce a new parameter such that \( e^x = z, \) then \( \omega = \omega(z) = y(\ln(z)), \) and rewrite the system (75)-(77) corresponding to this new parameter, then the system will be transformed to (85)-(87), subject to the initial conditions

\[ \omega_1(z)|_{z=e^v} = s_1, \quad \omega_2(z)|_{z=e^v} = s_2, \quad \text{and} \quad \omega_3(z)|_{z=e^v} = s_3, \]

then the solution of the image equation will have the form

\[ \omega_3(z; s_1, s_2, s_3) = \sum_{j=0}^{\infty} \frac{(z - e^v)^j}{j!} W_j(s_1, s_2, s_3). \quad (89) \]

(5) The coefficients of this image equation

\[ W_j(s_1, s_2, s_3) = \frac{1}{j!} \left[ \frac{1}{v} (s_1 D s_1 + (-s_1 - (s_1 - s_2^2)(s_2 + \frac{3}{8}))D s_2 + ((s_1 - s_2^2) \right \]

\[ (s_2 + \frac{3}{8}) D s_3)^j s_3, \quad j \in N_0 = \{0, 1, 2, ...\} \text{ have no Hankel rank} \]

\[ [Hr(\frac{W_j(s_1, s_2, s_3)}{j!}); \quad j \in N_0 = \{0, 1, 2, ...\} \text{ is unknown}. \]

So, the third component of the vector solution, \( y_3, \) cannot be expressed through finite sum or ratio of exponential functions also.
8. Conclusion and Remarks

In this article, we introduced the space of functions \( \mathcal{C} \) to be the space of consideration. We had constructed theoretical and numerical study for a system of fractional differential equations under the space of functions \( \mathcal{C} \), which is a subclass from the wider space of functions \( \mathcal{C} \) but it contains additional properties. We generalized the operator method introduced by Navickas to construct a general theorem for a system of ordinary differential equations. We provided an algorithm to construct an exact solution for ordinary fractional differential equations under the space of functions \( \mathcal{C} \). The criterion of the existence of the solution of this class of equations depend on the existence of Hankel rank of the produced sequence and conditioned by the roots of its characteristic equation. Several examples were used to illustrate the proposed concept. Our study can be extended comprising the other standard functions. Finally, we point out that the corresponding analytical computations are obtained according to the proposed technique by using Mathematica 6.

References


